POINTWISE BI-SLANT SUBMERSIONS WHOSE TOTAL MANIFOLDS ARE ALMOST PRODUCT RIEMANNIAN

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Abstract. In this project, we present the notion of a pointwise bi-slant Riemannian submersion in the almost product context, generalizing the ideas of slant, pointwise slant, anti-invariant, semi-slant, pointwise semi-slant, and bi-slant submersions. Necessary and sufficient conditions for the integrability and total geodesicity of certain distributions of the fibers of a pointwise bi-slant submersion are given. Moreover, we provide many relations between pluriharmonicity, ϕ-invariance, integrability, and total geodesicity for such submersions.

1. Introduction

The theory of submanifolds has been shown to be quite useful in Differential Geometry. It:

- generalizes the concept of curves and surfaces to higher dimensions,
- enables the study of complex geometries that Euclidean spaces cannot fully capture,
- provides a framework to analyze the intrinsic properties of curvature, tangent spaces, geodesics, and other geometric structures,
- allows for the representation of complex shapes and motion paths in an efficient, compact manner in robotics and computer graphics,
- develops powerful tools for shape matching, registration, and analysis by representing shapes as submanifolds in shape analysis,
- helps solve differential equations in various applications, such as fluid dynamics, heat conduction, and elasticity,
- aids in representing configuration spaces of physical systems,
- aids in understanding the underlying structure of high-dimensional data in terms of data visualization, dimensionality reduction, and clustering.

Overall, submanifolds provide a powerful and flexible framework for understanding complex geometries and their intrinsic properties. They offer a deeper insight into the structure of spaces, and crucially, they find applications across a wide range of disciplines, making them an essential concept in modern mathematics and its various applications.

The importance of submanifolds prompted the Geometers to define and study specific submanifolds. One of the ways to obtain a submanifold is by working with *submersions*. The most well-known and studied map of this kind is the *Riemannian Submersion*. The notion of Riemannian submersion was introduced first by O'Neill with the following definition.

Definition 1.1. [9] Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, where $\dim(M_1)$ is greater than $\dim(M_2)$. A surjective mapping

$$\Psi : (M_1, g_1) \to (M_2, g_2)$$

is called a *Riemannian submersion* if

(S1) Ψ has maximal rank, and

(S2) Ψ_*, restricted to $\ker \Psi_*^{\perp}$, is a linear isometry.

In this case, for each $q \in M_2$, $\Psi^{-1}(q)$ is a k-dimensional submanifold of M_1, called a *fiber*, where $k = \dim(M_1) - \dim(M_2)$.

Later, Watson considered Riemannian submersions between almost Hermitian manifolds and called them *almost Hermitian submersions* [23], where the submersion is now a complex mapping. Another submersion, called an *anti-invariant Riemannian submersion*, was defined also in a complex context by Şahin [12]: in this case, the fibers are horizontal under the action of the almost complex structure, i.e. they are anti-invariant submanifolds of the total space. Outside of these specific cases, the notion of a Riemannian submersion has been considered in many other contexts, such as contact [32], complex [5], almost product [33], and more. In all of these studies, submersions were defined based on the action of the structure of the manifold on the fibers.

Crucially for this report, another type of submersion called a *bi-slant Riemannian submersion* was given in a complex context with the following definition.

Definition 1.2. [26] Let (M, g, J) be a Kaehler manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, J) \to (N, g_N)$ is called a ***bi-slant submersion*** if there are two slant distributions $\mathcal{D}^{\theta_1} \subset ker\pi_*$ and $\mathcal{D}^{\theta_2} \subset ker\pi_*$ such that

$$ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}, \tag{1.1}$$

where $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ have slant angles θ_1 and θ_2, respectively. If each slant angle is neither zero nor $\frac{\pi}{2}$, then the bi-slant submersion is called a *proper bi-slant submersion*.

The angles θ_1 and θ_2 are constants in this definition. Generalizing, Sepet et al. considered the angles as functions and defined a *pointwise bi-slant Riemannian submersion* in a complex context [18] and a contact context [19]. However, since the structure of the manifold plays a role in defining a submersion, a natural question is:

What if we consider a pointwise bi-slant Riemannian submersion in an almost product context?

This is the very question this report considers. First, we present preliminary information to understand the coming results, including formally defining an almost product Riemannian manifold and a pointwise bi-slant Riemannian submersion, as well as presenting an example of a pointwise bi-slant Riemannian submersion in an almost product context.

2. Preliminaries

2.1. **Riemannian submersions.** When considering a Riemannian submersion $\pi : (M, g) \to (N, \bar{g})$, we recall the following observations and concepts:

- A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers.
- We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $\ker \pi_*$ and the horizontal distribution $\ker \pi_*^{\perp}$, respectively.
- The manifold (M, g) is called the *total manifold* and $(N, \bar{g})$ is called the *base manifold*.
- A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e.,

$$\pi_* X_p = X_{*\pi(p)}, \ \forall p \in M.$$

The last fact given above yields the following lemma [9], which explains the preservation of brackets, inner products, and covariant derivatives.

Lemma 2.1. *Let $\pi : (M, g) \to (N, \bar{g})$ be a Riemannian submersion between Riemannian manifolds. If X and Y are basic vector fields, then*

- $g(X, Y) = \bar{g}(X_*, Y_*) \circ \pi$,
- *the horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field corresponding to $[X_*, Y_*]$,*
- *the horizontal part $\mathcal{H}(\nabla^M_X Y)$ of $\nabla^M_X Y$ is the basic vector field corresponding to $\nabla^N_{X_*} Y_*$,*
- *$[U, X]$ is vertical for any vector field U of $\ker \pi_*$.*

2.2. **O'Neill's tensors.** The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E G = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}G, \tag{2.1}$$

$$\mathcal{A}_E G = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}G + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}G, \tag{2.2}$$

where E and G are vector fields of M and ∇ is the Levi-Civita connection of g. It is clear $\mathcal{T}$ and $\mathcal{A}$ reverse the vertical and horizontal distributions, respectively. We also see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of M, meaning for all $E, F, G \in \Gamma(TM)$,

$$g(F, \mathcal{T}_E G) = -g(\mathcal{T}_E F, G),$$
$$g(F, \mathcal{A}_E G) = -g(\mathcal{A}_E F, G).$$

Moreover, let V, W be vertical and X, Y be horizontal vector fields on M. Then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W \equiv \mathcal{V}\nabla_V W$. Lastly, if X is basic,

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V. \tag{2.9}$$

For more details, we refer to O'Neill's paper [9] and the book [5].
Remark: in this paper, we will assume all horizontal vector fields are basic. $\diamondsuit$

2.3. **Almost product Riemannian and locally product Riemannian manifolds.** An m-dimensional manifold M is called an *almost product manifold* if it is equipped with an *almost product structure* ϕ, which is a tensor field of type (1,1), satisfying

$$\phi^2 = id\ (\phi \neq \pm id). \tag{2.10}$$

We denote an almost product manifold by (M, ϕ). Moreover, if (M, ϕ) admits a Riemannian metric g satisfying

$$g(\phi E, \phi G) = g(E, G) \text{ for each } E, G \in \Gamma(TM), \tag{2.11}$$

then M is said to be an *almost product Riemannian manifold.*
Now, let ∇ be the Riemannian connection with respect to the metric g on M. Then M is called a *locally product Riemannian manifold* (briefly, *l.p.R.*) if ϕ is parallel with respect to the connection, i.e. [25]

$$\nabla\phi = 0. \tag{2.12}$$

3. Pointwise bi-slant submersions

With the relevant background covered, we are now able to define a pointwise bi-slant submersion in an almost product context. This submersion sets itself apart from the previously-defined bi-slant submersion (see [26]) since slant angles are not constant here but rather functions over the total manifold. The pointwise bi-slant submersion can therefore be seen as a generalization of the bi-slant submersion.

Definition 3.1. Let (M, g, ϕ) be an almost product Riemannian manifold and $(N, \bar{g})$ be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, \phi) \to (N, \bar{g})$ is called a *pointwise bi-slant Riemannian submersion* if the vertical distribution $\ker \pi_*$ decomposes into two orthogonal complementary distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$:

$$\ker \pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}. \tag{3.1}$$

In this case, $\mathcal{D}^{\theta_i}$ for $i \in \{1, 2\}$ is a pointwise slant distribution where the angle θ_i between ϕU and the space $(\mathcal{D}^{\theta_i})_q$ (for all $q \in M$) is independent of the choice of nonzero vector $U \in \Gamma(\mathcal{D}^{\theta_i})_q$. We call θ_1 and θ_2 the *slant functions of the pointwise bi-slant Riemannian submersion.*

Example. Consider the standard Euclidean space $\mathbb{R}^8$ with the standard metric g. One can see that

$$\phi_1(x_1, x_2, ..., x_8) = (-x_3, x_4, -x_1, x_2, -x_7, x_8, -x_5, x_6)$$

and

$$\phi_2(x_1, x_2, ..., x_8) = (x_2, x_1, x_4, x_3, x_6, x_5, x_8, x_7)$$

are almost product Riemannian structures on $\mathbb{R}^8$, where $\phi_1\phi_2 = -\phi_2\phi_1$. For any smooth function $\pi : \mathbb{R}^8 \to \mathbb{R}^4$, we can define a new almost product Riemannian structure given by

$$\phi_{1,2} = f\phi_1 + h\phi_2,$$

where f and h are defined by

$$\begin{aligned} f : \mathbb{R}^8 - \{-1\} &\to \mathbb{R}, \\ f(x_1, x_2, ..., x_8) &= -\frac{x_1}{\sqrt{(x_1)^2 + 1}}, \\ h : \mathbb{R}^8 &\to \mathbb{R}, \\ h(x_1, x_2, ..., x_8) &= \frac{1}{\sqrt{(x_1)^2 + 1}}. \end{aligned}$$

It is easy to check that $(\mathbb{R}^8, \phi_{1,2}, g)$ is an almost product Riemannian manifold.

Now, let π be a map between $\mathbb{R}^8$ and $\mathbb{R}^4$ defined by

$$\pi(x_1, x_2, ..., x_8) = \left(\frac{x_1 - x_3}{\sqrt{2}}, \frac{x_2 - x_4}{\sqrt{2}}, \frac{x_5 + x_8}{\sqrt{2}}, \frac{-x_6 + x_7}{\sqrt{2}} \right).$$

Then the decomposition $\ker \pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$ where

$$\begin{aligned} \mathcal{D}^{\theta_1} &= span\left\{ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} \right\}, \\ \mathcal{D}^{\theta_2} &= span\left\{ \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} \right\} \end{aligned}$$

shows that π is a pointwise bi-slant submersion with slant functions

$$\theta_1 = \cos^{-1}(g) \quad \text{and} \quad \theta_2 = \cos^{-1}(-f).$$

(These slant functions can be found by direct calculation.) ♢

Further analyzing the almost product structure ϕ, it is known that we may find ways to decompose the vector fields ϕX and $\phi\xi$, where $X \in \Gamma(\ker \pi_*)$ and $\xi \in \Gamma(\ker {\pi_*}^{\perp})$, into horizontal and vertical components. These decompositions are not unique, and they are in general different for horizontal and vertical fields. Thus, we notate them generally in the following definition.

Definition 3.2. Let $\pi : (M, g, \phi) \to (N, \bar{g})$ be a pointwise bi-slant submersion from an almost product Riemannian manifold M onto a Riemannian manifold N. Then, for any $V \in \Gamma(\ker \pi_*)$, we may decompose $\phi(V)$ as

$$\phi(V) = tV + nV, \tag{3.2}$$

where $tV \in \Gamma(\ker \pi_*)$ and $nV \in \Gamma(\ker {\pi_*}^{\perp})$. Similarly, for any $\xi \in \Gamma(\ker {\pi_*}^{\perp})$,

$$\phi\xi = \mathscr{T}\xi + N\xi, \tag{3.3}$$

where $\mathscr{T}\xi \in \Gamma(\ker \pi_*)$ and $N\xi \in \Gamma(\ker {\pi_*}^{\perp})$.

With these canonical forms now written, we may find immediate corollaries that will be used often in the coming theorems.

Corollary 3.3. *Let $\pi : (M, \phi, g) \to (N, \bar{g})$ be a pointwise bi-slant Riemannian submersion from almost product Riemannian manifold M onto Riemannian manifold N. Then, for any $X, Y \in \Gamma(\ker \pi_*)$ and any $\beta, \xi \in \Gamma(\ker {\pi_*}^{\perp})$,*

I) $g(tX, Y) = g(X, tY)$,

II) $g(N\beta, \xi) = g(\beta, N\xi)$.

Corollary 3.4. *Let $\pi : (M, g, \phi) \to (N, \bar{g})$ be a pointwise bi-slant submersion from almost product Riemannian manifold M onto Riemannian manifold N. Then, for any $X \in \Gamma(\ker \pi_*)$ and any $\xi \in \Gamma(\ker {\pi_*}^\perp)$,*

I) *i)* $X = t^2X + \mathscr{T}nX$,

ii) $0 = NnX + ntX$,

II) *i)* $\xi = N^2\xi + n\mathscr{T}\xi$,

ii) $0 = t\mathscr{T}\xi + \mathscr{T}N\xi$.

Proof. Let $X \in \ker \pi_*$. Then:

$$\begin{aligned} X &= \phi^2 X \\ &= \phi(tX + nX) \\ &= t^2X + ntX + \mathscr{T}nX + NnX \\ \Rightarrow 0 &= (-X + t^2X + \mathscr{T}nX) + (NnX + ntX). \end{aligned}$$

We have divided the above sum into horizontal and vertical parts. Since both parts sum to 0, both must individually be 0, yielding item I(i) and item I(ii). The other two may be found in a similar manner. □

Corollary 3.5. *Let $\pi : (M, g, \phi) \to (N, \bar{g})$ be a pointwise bi-slant Riemannian submersion from locally product Riemannian manifold M to Riemannian manifold N where ∇ is the Levi-Civita connection of g. Then, for any $X, Y \in \Gamma(\ker \pi_*)$ and any $\beta, \xi \in \Gamma(\ker {\pi_*}^\perp)$,*

I) *i)* $\hat{\nabla}_X tY + \mathcal{T}_X(nY) = \mathscr{T}\mathcal{T}_X(Y) + t\hat{\nabla}_X Y$

ii) $\mathcal{T}_X(tY) + \mathcal{A}_{nY}(X) = N\mathcal{T}_X(Y) + n\hat{\nabla}_X Y$,

II) *i)* $\hat{\nabla}_\beta \mathscr{T}\xi + \mathcal{A}_\beta(N\xi) = \mathscr{T}\mathcal{H}\nabla_\beta\xi + t\mathcal{A}_\beta(\xi)$

ii) $\mathcal{A}_\beta(\mathscr{T}\xi) + \mathcal{H}\nabla_\beta N\xi = N\mathcal{H}\nabla_\beta\xi + n\mathcal{A}_\beta(\xi)$,

III) *i)* $\hat{\nabla}_\beta tX + \mathcal{A}_\beta(nX) = \mathscr{T}\mathcal{A}_\beta(X) + t\hat{\nabla}_\beta X$

ii) $\mathcal{A}_\beta(tX) + \mathcal{H}\nabla_\beta nX = N\mathcal{A}_\beta(X) + n\hat{\nabla}_\beta X$,

IV) *i)* $\hat{\nabla}_X \mathscr{T}\xi + \mathcal{T}_X(N\xi) = \mathscr{T}\mathcal{A}_\xi(X) + t\mathcal{T}_X(\xi)$

ii) $\mathcal{T}_X(\mathscr{T}\xi) + \mathcal{A}_{N\xi}(X) = N\mathcal{A}_\xi(X) + n\mathcal{T}_X(\xi)$.

Proof. Let $X, Y \in \Gamma(\ker \pi_*)$. Because ϕ is parallel to the connection ∇, we know $\nabla_X \phi Y = \phi\nabla_X Y$. Then we may use eq. (2.5) and eq. (2.6) to show the following:

$$\begin{aligned} \nabla_X \phi Y &= \phi\nabla_X Y \\ \nabla_X tY + \nabla_X nY &= \phi(\mathcal{T}_X(Y) + \hat{\nabla}_X Y) \\ \mathcal{T}_X(tY) + \hat{\nabla}_X tY + \mathcal{T}_X(nY) + \mathcal{H}\nabla_X nY &= \mathscr{T}\mathcal{T}_X(Y) + N\mathcal{T}_X(Y) + t\hat{\nabla}_X Y + n\hat{\nabla}_X Y. \end{aligned}$$

Each vector field in the final equation is either horizontal or vertical. Thus, we may rewrite this equation into a sum of vertical vector fields plus a sum of horizontal vector fields. These sums add to 0, and so the horizontal and vertical parts must both be 0 individually. This immediately yields item I(i) and, upon substitution of $\mathcal{H}\nabla_X nY = \mathcal{A}_{nY}(X)$ from eq. (2.9), we find item I(ii). All other equations may be found using a similar method. □

It is known in the literature that the pointwise slant distributions $\mathcal{D}^{\theta_j}$ ($j \in \{1, 2\}$) are t-invariant, meaning for all $V \in \Gamma(\mathcal{D}^{\theta_j})$, $tV \in \Gamma(\mathcal{D}^{\theta_j})$. Using this fact, the next corollary follows.

Corollary 3.6. *Let $\pi : (M, g, \phi) \to (N, \bar{g})$ be a pointwise bi-slant Riemannian submersion from almost product Riemannian manifold M onto Riemannian manifold N, and let $V \in \Gamma(\mathcal{D}^{\theta_i})$ where $i \in \{1, 2\}$ Then,*

I) $t^2V = \cos^2(\theta_i)V$,

II) $\mathscr{T} nV = \sin^2(\theta_i)V$,

III) $g(tV, tV) = \cos^2(\theta_i)g(V, V)$,

IV) $g(nV, nV) = \sin^2(\theta_i)g(V, V)$.

Proof. Without loss of generality, let $V \in \Gamma(\mathcal{D}^{\theta_1})$.

I) Since $tV \in \Gamma(\mathcal{D}^{\theta_1})$, we get that

$$\begin{aligned}\cos\theta_1 &= \frac{g(\phi V, tV)}{||V||\,||tV||}\\ &= \frac{g(tV, tV)}{||V||\,||tV||}\\ &= \frac{||tV||}{||V||}.\end{aligned}$$

But then, because $g(\phi V, tV) = g(V, \phi tV)$, we also have that

$$\begin{aligned}\cos\theta_1 &= \frac{g(V, \phi tV)}{||V||\,||tV||}\\ &= \frac{g(V, t^2V + ntV)}{||V||\,||tV||}\\ &= \frac{g(V, t^2V)}{||V||\,||tV||}.\end{aligned}$$

Which, in turn, implies:

$$\begin{aligned}\cos^2(\theta_1) = \frac{g(V, t^2V)}{||V||^2} &\implies g(V, V)\cos^2\theta_1 = g(V, t^2V)\\ &\implies t^2V = \cos^2(\theta_1)V.\end{aligned}$$

II) Using item I(i) from corollary 3.4, the result follows immediately.

III) Using item I) from corollary 3.3 and result I from this lemma, the result follows.

IV) Notice that:

$$\begin{aligned}g(V, V) &= g(\phi V, \phi V)\\ &= g(tV, tV) + g(nV, nV)\\ &= \cos^2\theta_1 g(V, V) + g(nV, nV),\\ \Rightarrow g(nV, nV) &= \sin^2(\theta_1)g(V, V).\end{aligned}$$

□

We may also draw more conclusions on the behavior of the canonical forms $t, n, \mathscr{T}$, and N using their derivatives. They are given below, and the coming theorem relates the parallel conditions of these derivatives.

Definition 3.7. Let π be a Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Then the derivatives of $t, n, \mathscr{T}$, and N are given by the following: for all $U, V \in \Gamma(\ker\pi_*)$ and $\xi \in \Gamma(\ker{\pi_*}^\perp)$,

$$\begin{aligned}(\nabla_U t)V &= \hat{\nabla}_U tV - t\hat{\nabla}_U V,\\ (\nabla_U n)V &= \mathcal{H}\,\nabla_U nV - n\hat{\nabla}_U V,\\ (\nabla_U \mathscr{T})\xi &= \hat{\nabla}_U \mathscr{T}\xi - \mathscr{T}\,\mathcal{H}\,\nabla_U \xi,\\ (\nabla_U N)\xi &= \mathcal{H}\,\nabla_U N\xi - N\,\mathcal{H}\,\nabla_U \xi.\end{aligned}$$

Using eq. (2.9) and corollary 3.5, we may rewrite these derivatives in other forms. These are given below:

$$(\nabla_U t)V = \hat{\nabla}_U tV - t\hat{\nabla}_U V = \mathscr{T}\mathcal{T}_U(V) - \mathcal{T}_U(nV), \tag{3.4}$$

$$(\nabla_U n)V = \mathcal{A}_{nV}(U) - n\hat{\nabla}_U V = N\mathcal{T}_U(V) - \mathcal{T}_U(tV), \tag{3.5}$$

$$(\nabla_U \mathscr{T})\xi = \hat{\nabla}_U \mathscr{T}\xi - \mathscr{T}\mathcal{A}_\xi(U) = t\mathcal{T}_U(\xi) - \mathcal{T}_U(N\xi), \tag{3.6}$$

$$(\nabla_U N)\xi = \mathcal{A}_{N\xi}(U) - N\mathcal{A}_\xi(U) = n\mathcal{T}_U(\xi) - \mathcal{T}_U(\mathscr{T}\xi). \tag{3.7}$$

Theorem 3.8. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) onto a Riemannian manifold $(N, \bar{g})$. Then $\nabla n = 0$ if and only if $\nabla \mathscr{T} = 0$.*

Proof. Let $U, V \in \Gamma(\ker \pi_*)$ and $\xi \in \Gamma(\ker {\pi_*}^{\perp})$. Then we know the following is true:

$$g(V, t\mathcal{T}_U(\xi)) = g(tV, \mathcal{T}_U(\xi)) = -g(\mathcal{T}_U(tV), \xi).$$

If we let $\nabla n = 0$, then we know from eq. (3.5) that $\mathcal{T}_U(tV) = N\mathcal{T}_U(V)$. Therefore,

$$g(V, t\mathcal{T}_U(\xi)) = -g(N\mathcal{T}_U(V), \xi) = -g(\mathcal{T}_U(V), N\xi) = g(V, \mathcal{T}_U(N\xi)),$$
$$\implies g(V, t\mathcal{T}_U(\xi) - \mathcal{T}_U(N\xi)) = 0.$$

This implies $t\mathcal{T}_U(\xi) - \mathcal{T}_U(N\xi) = 0$ for all vertical U and horizontal ξ; therefore, eq. (3.6) implies $\nabla \mathscr{T} = 0$. We can then employ a similar method to prove $\nabla \mathscr{T} = 0 \implies \nabla n = 0$. First, we know

$$g(\xi, N\mathcal{T}_U(V)) = g(N\xi, \mathcal{T}_U(V)) = -g(\mathcal{T}_U(N\xi), V).$$

Then, assuming $\nabla \mathscr{T} = 0$, we know $\mathcal{T}_U(N\xi) = t\mathcal{T}_U(\xi)$ by eq. (3.6). Therefore,

$$g(\xi, N\mathcal{T}_U(V)) = -g(t\mathcal{T}_U(\xi), V) = -g(\mathcal{T}_U(\xi), tV) = g(\xi, \mathcal{T}_U(tV))$$
$$\implies g(\xi, N\mathcal{T}_U(V) - \mathcal{T}_U(tV)) = 0.$$

This implies $N\mathcal{T}_U(V) - \mathcal{T}_U(tV) = 0$ for each $U, V \in \Gamma(\ker \pi_*)$, meaning $\nabla n = 0$. □

4. Integrability

Using these identities with the canonical structures, we may now discuss properties of the distributions yielded from pointwise bi-slant Riemannian submersions. The first condition of interest is integrability: a distribution of a manifold is *integrable* if, for any point on the manifold, there exists a submanifold containing the point such that the tangent space at the point is equal to the distribution at the point. Amazingly, Ferdinand Georg Frobenius was able to show that a distribution D of a smooth manifold is integrable if and only if for any $X, Y \in \Gamma(D)$, $[X, Y] \in \Gamma(D)$. For our specific work, we derive some equivalent conditions for the integrability of the distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$. To do this, we first define the second fundamental form of a smooth map between manifolds.

Definition 4.1. Let π be a C^∞ map between a Riemannian manifold (M, g) and a Riemannian manifold $(N, \bar{g})$. The *second fundamental form* of π is then given by

$$(\nabla \pi_*)(X, Y) = \nabla^\pi_X \pi_* Y - \pi_*(\nabla_X Y), \tag{4.1}$$

where ∇^π is the pullback connection and we, conveniently, denote ∇ as the Levi-Civita connections of g and $\bar{g}$. [27]

We want to incorporate the metric g into our equivalent conditions for the integrability of the slant distributions. Thus, to give these conditions, we need the following result.

Theorem 4.2. *Let $\pi : (M, g, \phi) \to (N, \bar{g})$ be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold M to a Riemannian manifold N where $\ker \pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$, and let $i, j \in \{1, 2\}$ with $i \neq j$. Then, for any $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$,*

I) $g(\nabla_X Y, U) = -\csc^2 \theta_j \Big[g(X, \mathcal{T}_Y(ntU) + \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \Big]$

II) $g(\nabla_X Y, U) = \sec^2 \theta_j \Big[g(\hat{\nabla}_X tY, tU) + g(X, \mathcal{T}_{tU}(nY) + \mathcal{T}_Y(ntU)) \Big]$

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$. Then,

$$\begin{aligned} g(\nabla_X Y, U) = g(\phi\nabla_X Y, \phi U) &= g(\phi\nabla_X Y, tU) + g(\phi\nabla_X Y, nU) \\ &= g(\phi^2\nabla_X Y, \phi tU) + g(\phi\nabla_X Y, nU) \\ &= g(\nabla_X Y, t^2 U) + g(\nabla_X Y, ntU) + g(\phi\nabla_X Y, nU). \end{aligned}$$

Using corollary 3.6, $t^2 U = \cos^2\theta_j U$. Thus:

$$\sin^2(\theta_j) g(\nabla_X Y, U) = g(\nabla_X Y, ntU) + g(\phi\nabla_X Y, nU). \tag{4.2}$$

We now manipulate the term $g(\phi\nabla_X Y, nU)$ using the parallel condition for ϕ, the skew-symmetric properties of $\mathcal{T}$ and $\mathcal{A}$, equations 2.5 and 2.6, and the assumption that nY is basic:

$$\begin{aligned} g(\phi\nabla_X Y, nU) &= g(\nabla_X(\phi Y), nU) \\ &= g(\nabla_X(tY), nU) + g(\nabla_X(nY), nU) \\ &= g(\mathcal{T}_X(tY) + \hat{\nabla}_X tY, nU) + g(\mathcal{H}\,\nabla_X nY + \mathcal{T}_X(nY), nU) \\ &= g(\mathcal{T}_X(tY), nU) + g(\mathcal{H}\,\nabla_X nY, nU) \\ &= g(\mathcal{T}_{tY}(X), nU) + g(\mathcal{A}_{nY}(X), nU) \\ &= -g(X, \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \end{aligned}$$

We may now plug in this result for $g(\phi\nabla_X Y, nU)$ into eq. (4.2), and then use the same properties and lemmas as used previously to derive the following:

$$\begin{aligned} \sin^2(\theta_j) g(\nabla_X Y, U) &= g(\nabla_X Y, ntU) - g(X, \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \\ &= g(\mathcal{T}_Y(X), ntU) - g(X, \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \\ &= -g(X, \mathcal{T}_Y(ntU) + \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)). \end{aligned}$$

This implies the first result.

To find the second result, we use the same relation:

$$\begin{aligned} g(\nabla_X Y, U) &= g(\phi\nabla_X Y, tU) + g(\phi\nabla_X Y, nU) \\ &= g(\phi\nabla_X Y, tU) + g(\phi^2\nabla_X Y, \phi nU) \\ &= g(\phi\nabla_X Y, tU) + g(\nabla_X Y, \mathscr{T} nU) + g(\nabla_X Y, NnU). \end{aligned}$$

Using item I(i) in corollary 3.4, we may rewrite the middle term as follows:

$$\begin{aligned} g(\nabla_X Y, U) &= g(\phi\nabla_X Y, tU) + g(\nabla_X Y, U - t^2 U) + g(\nabla_X Y, NnU) \\ &= g(\phi\nabla_X Y, tU) + g(\nabla_X Y, U) - \cos^2(\theta_j) g(\nabla_X Y, U) + g(\nabla_X Y, NnU) \\ \implies \cos^2(\theta_j) g(\nabla_X Y, U) &= g(\phi\nabla_X Y, tU) + g(\nabla_X Y, NnU) \end{aligned}$$

Using the parallel condition, the corollaries for the canonical forms, and similar methods used to find the first result, we may rewrite the equation above as follows:

$$\begin{aligned} \cos^2(\theta_j) g(\nabla_X Y, U) &= g(\nabla_X \phi Y, tU) + g(\nabla_X Y, NnU) \\ &= g(\nabla_X tY, tU) + g(\nabla_X nY, tU) + g(\nabla_X Y, NnU) \\ &= g(\mathcal{T}_X(tY) + \hat{\nabla}_X tY, tU) + g(\mathcal{H}\,\nabla_X nY + \mathcal{T}_X(nY), tU) + g(\mathcal{T}_X(Y) + \hat{\nabla}_X Y, NnU) \\ &= g(\hat{\nabla}_X tY, tU) + g(\mathcal{T}_X(nY), tU) + g(\mathcal{T}_X(Y), NnU) \\ &= g(\hat{\nabla}_X tY, tU) - g(\mathcal{T}_X(tU), nY) + g(\mathcal{T}_Y(X), NnU) \\ &= g(\hat{\nabla}_X tY, tU) - g(\mathcal{T}_{tU}(X), nY) - g(\mathcal{T}_Y(NnU), X) \\ &= g(\hat{\nabla}_X tY, tU) + g(\mathcal{T}_{tU}(nY), X) - g(\mathcal{T}_Y(NnU), X) \\ &= g(\hat{\nabla}_X tY, tU) + g(\mathcal{T}_{tU}(nY) - \mathcal{T}_Y(NnU), X) \\ &= g(\hat{\nabla}_X tY, tU) + g(\mathcal{T}_{tU}(nY) + \mathcal{T}_Y(ntU), X). \end{aligned}$$

This yields the second result. □

Using this important result, we are now ready to give conditions for the integrability of the slant distributions.

Theorem 4.3. *Let π be a pointwise bi-slant Riemannian submersion mapping from an l.p.R manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$ where $\ker \pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$. Let $i, j \in \{1, 2\}$ with $i \neq j$. Then the following conditions are equivalent: for all $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$,*

I) $\mathcal{D}^{\theta_i}$ *is integrable,*

II) $0 = g(nU, \mathcal{T}_{tX}(Y) + \mathcal{A}_{nX}(Y) - (\mathcal{T}_{tY}(X) + \mathcal{A}_{nY}(X)))$.

III) $0 = g(tU, \hat{\nabla}_X tY - \hat{\nabla}_Y tX) + \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) - \bar{g}(\pi_*(nX), (\nabla\pi_*)(Y, tU))$

IV) $0 = \bar{g}(\pi_*(nU), (\nabla\pi_*)(tX, Y) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(tY, X) - (\nabla\pi_*)(nY, X))$

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$. We may use the Frobenius theorem to show that $\mathcal{D}^{\theta_i}$ is integrable if and only if $g([X, Y], U) = 0$ for all $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$. This will be used in proving the equivalence of the four conditions.

To show $(I) \iff (II)$, we take equation 1 from theorem 4.2 to yield the following:

$$\begin{aligned}
g([X, Y], U) &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\
&= -\csc^2\theta_j[g(X, \mathcal{T}_Y(ntU) + \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \\
&\quad - g(Y, \mathcal{T}_X(ntU) + \mathcal{T}_{tX}(nU) + \mathcal{A}_{nX}(nU))] \\
&= -\csc^2\theta_j[g(X, \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) - g(Y, \mathcal{T}_{tX}(nU) + \mathcal{A}_{nX}(nU)) \\
&\quad + g(X, \mathcal{T}_Y(ntU)) - g(Y, \mathcal{T}_X(ntU))] \\
&= -\csc^2\theta_j[g(X, \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) - g(Y, \mathcal{T}_{tX}(nU) + \mathcal{A}_{nX}(nU)) + 0] \\
&= \csc^2\theta_j[g(\mathcal{T}_{tY}(X), nU) + g(\mathcal{A}_{nY}(X), nU) - (g(\mathcal{T}_{tX}(Y), nU) + g(\mathcal{A}_{nX}(Y), nU))] \\
&= -\csc^2(\theta_j) g(nU, \mathcal{T}_{tX}(Y) + \mathcal{A}_{nX}(Y) - (\mathcal{T}_{tY}(X) + \mathcal{A}_{nY}(X))).
\end{aligned} \tag{4.3}$$

Equivalence is then clear from the final equation.

To prove $(I) \iff (III)$, consider equation 2 of theorem 4.2. Then:

$$\begin{aligned}
g([X, Y], U) &= \sec^2\theta_j[g(\hat{\nabla}_X tY - \hat{\nabla}_Y tX, tU) + g(X, \mathcal{T}_{tU}(nY) + \mathcal{T}_Y(ntU)) \\
&\quad - g(Y, \mathcal{T}_{tU}(nX) + \mathcal{T}_X(ntU))] \\
&= \sec^2\theta_j[g(\hat{\nabla}_X tY - \hat{\nabla}_Y tX, tU) - g(\mathcal{T}_X(tU), nY) - g(\mathcal{T}_Y(X), ntU) \\
&\quad + g(\mathcal{T}_Y(tU), nX) + g(\mathcal{T}_X(Y), ntU)] \\
&= \sec^2\theta_j[g(\hat{\nabla}_X tY - \hat{\nabla}_Y tX, tU) - g(\mathcal{T}_X(tU), nY) + g(\mathcal{T}_Y(tU), nX)].
\end{aligned}$$

We know $\mathcal{T}_X(tU)$ and $\mathcal{T}_Y(tU)$ are horizontal, and therefore we may use the fact that π is an isometry when restricted on $\ker \pi_*{}^\perp$ to yield the following:

$$\begin{aligned}
g([X, Y], U) &= \sec^2\theta_j[g(tU, \hat{\nabla}_X tY - \hat{\nabla}_Y tX) - \bar{g}(\pi_*(\mathcal{T}_X(tU)), \pi_*(nY)) \\
&\quad + \bar{g}(\pi_*(\mathcal{T}_Y(tU)), \pi_*(nX))] \\
&= \sec^2\theta_j[g(tU, \hat{\nabla}_X tY - \hat{\nabla}_Y tX) - \bar{g}(\pi_*(\nabla_X tU), \pi_*(nY)) \\
&\quad + \bar{g}(\pi_*(\nabla_Y tU), \pi_*(nX))].
\end{aligned}$$

Let us then use the definition of the second fundamental form of π to rewrite the push-forwards above:

$$\begin{aligned}
(\nabla\pi_*)(Y, tU) &= \nabla^\pi_Y \pi_* tU - \pi_*(\nabla_Y tU) \\
&= \nabla^\pi_Y 0 - \pi_*(\nabla_Y tU) \\
&= -\pi_*(\nabla_Y tU) \\
\implies \pi_*(\nabla_Y tU) &= -(\nabla\pi_*)(Y, tU).
\end{aligned}$$

Similarly, $\pi_*(\nabla_X tU) = -(\nabla\pi_*)(X, tU)$, and thus:

$$\begin{aligned} g([X,Y],U) = \sec^2\theta_j[&g(tU, \hat{\nabla}_X tY - \hat{\nabla}_Y tX) + \bar{g}((\nabla\pi_*)(X,tU), \pi_*(nY)) \\ &- \bar{g}((\nabla\pi_*)(Y,tU), \pi_*(nX))]. \end{aligned}$$

To show I and IV are equivalent, consult eq. (4.3) and note that all terms in the metric are horizontal. We may then use the fact that π is an isometry on $\ker\pi_*^{\perp}$ to yield the following:

$$\begin{aligned} g([X,Y],U) &= -\csc^2(\theta_j)\bar{g}(\pi_*(nU), \pi_*(\mathcal{T}_{tX}(Y)) + \pi_*(\mathcal{A}_{nX}(Y)) - \pi_*(\mathcal{T}_{tY}(X)) - \pi_*(\mathcal{A}_{nY}(X))) \\ &= -\csc^2(\theta_j)\bar{g}(\pi_*(nU), \pi_*(\mathcal{H}\nabla_{tX}Y) + \pi_*(\mathcal{H}\nabla_{nX}Y) - \pi_*(\mathcal{H}\nabla_{tY}X) - \pi_*(\mathcal{H}\nabla_{nY}X)) \\ &= -\csc^2(\theta_j)\bar{g}(\pi_*(nU), \pi_*(\nabla_{tX}Y) + \pi_*(\nabla_{nX}Y) - \pi_*(\nabla_{tY}X) - \pi_*(\nabla_{nY}X)). \end{aligned}$$

In a similar manner as when we proved $(I) \iff (III)$, we may rewrite the final equation above using the second fundamental form of π:

$$g([X,Y],U) = \csc^2(\theta_j)\bar{g}(\pi_*(nU), (\nabla\pi_*)(tX,Y) + (\nabla\pi_*)(nX,Y) - (\nabla\pi_*)(tY,X) - (\nabla\pi_*)(nY,X)).$$

□

5. Totally Geodesic Distributions

We now examine the totally geodesic condition for the integral manifolds of the distributions $\ker\pi_*$, $\ker\pi_*^{\perp}$, $\mathcal{D}^{\theta_1}$, and $\mathcal{D}^{\theta_2}$. To define both integral manifolds and total geodesicity more clearly, we define an *integral manifold* of a distribution as a family of integral curves of all vector fields of that distribution. We also say a submanifold M_s of a Riemannian manifold (M, g) is *totally geodesic* if any geodesic on M_s using the Riemannian metric g restricted on the submanifold is also a geodesic on M. When describing this condition, we will use language like: '$\ker\pi_*$ is totally geodesic,' meaning that the integral submanifold corresponding to $\ker\pi_*$ is totally geodesic. Equivalently, we may also say that $\ker\pi_*$ defines totally geodesic foliations on the total manifold, and the same language applies for $\ker\pi_*^{\perp}$. For $\mathcal{D}^{\theta_j}$, the language: '$\mathcal{D}^{\theta_j}$ is totally geodesic' refers to the integral submanifold corresponding to $\mathcal{D}^{\theta_j}$ being totally geodesic to the integral manifold of $\ker\pi_*$.

The first equivalent conditions for the distributions $\ker\pi_*, \ker\pi_*^{\perp}, \mathcal{D}^{\theta_1}$, and $\mathcal{D}^{\theta_2}$ to be totally geodesic are given in the lemma below. These conditions will be used throughout this report.

Lemma 5.1. [26] *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$ with $\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$. Then, we have the following:*

I) $\ker\pi_*$ is totally geodesic if and only if $\mathcal{H}\nabla_X Y = 0$ for each $X, Y \in \Gamma(\ker\pi_*)$.

II) $\ker\pi_*^{\perp}$ is totally geodesic if and only if $\hat{\nabla}_Z W = 0$ for each $Z, W \in \Gamma(\ker\pi_*^{\perp})$.

III) $\mathcal{D}^{\theta_1}$ is totally geodesic if and only if $\hat{\nabla}_{X_1} Y_1 \in \Gamma(\mathcal{D}^{\theta_1})$ for each $X_1, Y_1 \in \Gamma(\mathcal{D}^{\theta_1})$.

IV) $\mathcal{D}^{\theta_2}$ is totally geodesic if and only if $\hat{\nabla}_{X_2} Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$ for each $X_2, Y_2 \in \Gamma(\mathcal{D}^{\theta_2})$.

From this lemma, two immediate corollaries follow. First, we see it is clear $\ker\pi_*$ is integrable if it is totally geodesic since, by assumption, $\hat{\nabla}_X Y = \nabla_X Y$ for each $X, Y \in \Gamma(\ker\pi_*)$. Therefore, $[X,Y] \in \Gamma(\ker\pi_*)$ for all vertical X and Y, implying $\ker\pi_*$ is integrable. By a similar procedure, $\ker\pi_*^{\perp}$ is integrable if $\ker\pi_*^{\perp}$ is totally geodesic. The first corollary of lemma 5.1 then shows a similar result also holds for the pointwise slant distributions $\mathcal{D}^{\theta_j}$:

Corollary 5.2. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$ with $\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$, and let $j \in \{1, 2\}$. Then $\mathcal{D}^{\theta_j}$ is integrable if $\mathcal{D}^{\theta_j}$ is totally geodesic.*

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$ and assume $\mathcal{D}^{\theta_j}$ is totally geodesic. Since X and Y are vertical, $[X, Y]$ is vertical, implying

$$[X,Y] = \mathcal{V}[X,Y] = (\hat{\nabla}_X Y - \hat{\nabla}_Y X) \in \Gamma(\mathcal{D}^{\theta_j}).$$

□

The second immediate corollary of lemma 5.1 gives conditions for when $\ker\pi_*$ and $\ker\pi_*^{\perp}$ are totally geodesic, as well as other useful relations when the totally geodesic condition is satisfied.

Corollary 5.3. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M,ϕ,g) to a Riemannian manifold $(N,\bar{g})$, and let $X,Y\in\Gamma(\ker\pi_*)$ and $\beta,\xi\in\Gamma(\ker{\pi_*}^{\perp})$. Then,*

I) $\ker\pi_*$ *is totally geodesic*
- *i) iff* $\mathcal{T}=0$ *on* $\ker\pi_*$,
- *ii) implies* $\mathcal{T}_X(nY)=t\nabla_XY-\nabla_XtY$,
- *iii) implies* $\mathcal{A}_{nY}(X)=n\nabla_XY$.

II) $\ker{\pi_*}^{\perp}$ *is totally geodesic*
- *i) iff* $\mathcal{A}=0$ *on* $\ker{\pi_*}^{\perp}$,
- *ii) implies* $\mathcal{A}_\beta(\mathscr{T}\xi)=N\nabla_\beta\xi-\nabla_\beta N\xi$,
- *iii) implies* $\hat{\nabla}_\beta\mathscr{T}\xi=\mathscr{T}\nabla_\beta\xi$.

Proof. Let $X,Y\in\Gamma(\ker\pi_*)$. First, from eq. (2.5) we know that

$$\nabla_XY=\mathcal{T}_X(Y)+\hat{\nabla}_XY,$$

and thus we can see that $\nabla_XY=\hat{\nabla}_XY \iff \mathcal{T}_X(Y)=0$. This shows item I(i).

Now, quote item I(i) from corollary 3.5:

$$\hat{\nabla}_XtY+\mathcal{T}_X(nY)=\mathscr{T}\mathcal{T}_X(Y)+t\hat{\nabla}_XY.$$

Assuming $\ker\pi_*$ is totally geodesic, we know $\mathcal{T}_X(Y)=0$, implying that $\mathscr{T}\mathcal{T}_X(Y)=0$. Using the same rule, $\hat{\nabla}_XtY=\nabla_XtY$ and $t\hat{\nabla}_XY=t\nabla_XY$. Therefore,

$$\mathcal{T}_X(nY)=t\nabla_XY-\nabla_XtY.$$

This proves item I(ii). For the third relation, we use the following equation from corollary 3.5:

$$\mathcal{T}_X(tY)+\mathcal{A}_{nY}(X)=N\mathcal{T}_X(Y)+n\hat{\nabla}_XY.$$

By assumption, we know $\mathcal{T}_X(tY)=0$ and $N\mathcal{T}_X(Y)=N(0)=0$. Therefore, after noting that $\hat{\nabla}_XY=\nabla_XY$, we know

$$\mathcal{A}_{nY}(X)=n\nabla_XY.$$

We can prove the other three equations with similar methods using eq. (2.8) and corollary 3.5. □

With these smaller corollaries finished, we may now provide theorems for equivalent conditions for $\ker\pi_*$ and $\ker{\pi_*}^{\perp}$ to be totally geodesic, which will lead to a larger theorem using both.

Theorem 5.4. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M,g,ϕ) to a Riemannian manifold $(N,\bar{g})$. Then the following are equivalent: for each $X,Y\in\Gamma(\ker\pi_*)$,*

I) $\ker\pi_*$ *is totally geodesic,*

II) $N\mathcal{T}_X(tY)+n\hat{\nabla}_XtY+n\mathcal{T}_X(nY)+N\mathcal{A}_{nY}(X)=0$,

III) $\nabla_XY=\mathscr{T}\mathcal{T}_X(tY)+t\hat{\nabla}_XtY+t\mathcal{T}_X(nY)+\mathscr{T}\mathcal{A}_{nY}(X)$,

Proof. Let $X,Y\in\Gamma(\ker\pi_*)$. Then,

$$\begin{aligned}\nabla_XY&=\phi\nabla_X\phi Y\\&=\phi(\nabla_XtY+\nabla_XnY)\\&=\phi(\mathcal{T}_X(tY)+\hat{\nabla}_XtY+\mathcal{T}_X(nY)+\mathcal{H}\nabla_XnY)\\&=\mathscr{T}\mathcal{T}_X(tY)+N\mathcal{T}_X(tY)+t\hat{\nabla}_XtY+n\hat{\nabla}_XtY\\&\qquad+t\mathcal{T}_X(nY)+n\mathcal{T}_X(nY)+\mathscr{T}(\mathcal{H}\nabla_XnY)+N(\mathcal{H}\nabla_XnY).\end{aligned}$$

This implies

$$\hat{\nabla}_X Y = \mathscr{T}\mathcal{T}_X(tY) + t\hat{\nabla}_X tY + t\mathcal{T}_X(nY) + \mathscr{T}\mathcal{A}_{nY}(X), \tag{5.1}$$
$$\mathcal{H}\nabla_X Y = N\mathcal{T}_X(tY) + n\hat{\nabla}_X tY + n\mathcal{T}_X(nY) + N\mathcal{A}_{nY}(X).$$

We know $\ker\pi_*$ is totally geodesic if and only if $\mathcal{H}\,\nabla_X Y = 0$ by lemma 5.1, and thus $N\mathcal{T}_X(tY)+n\hat{\nabla}_X tY + n\mathcal{T}_X(nY)+N\mathcal{A}_{nY}(X) = 0$ if and only if $\ker\pi_*$ is totally geodesic, showing $I \iff II$. To prove $I \iff III$, we know by lemma 5.1 again that $\ker\pi_*$ is totally geodesic if and only if $\hat{\nabla}_X Y = \nabla_X Y$, and so the first equation above shows I and III are equivalent. □

Theorem 5.5. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, g, ϕ) to a Riemannian manifold $(N, \bar{g})$. Then the following are equivalent: for all $\beta, \xi \in \Gamma(\ker{\pi_*}^\perp)$,*

I) $\ker{\pi_*}^\perp$ *is totally geodesic,*

II) $\mathscr{T}\mathcal{A}_\beta(\mathscr{T}\xi) + t\hat{\nabla}_\beta\mathscr{T}\xi + t\mathcal{A}_\beta(N\xi) + \mathscr{T}\,\mathcal{H}\,\nabla_\beta N\xi = 0.$

III) $\nabla_\beta\xi = N\mathcal{A}_\beta(\mathscr{T}\xi) + n\hat{\nabla}_\beta\mathscr{T}\xi + n\mathcal{A}_\beta(N\xi) + N\,\mathcal{H}\,\nabla_\beta N\xi$

Proof. Let $\beta, \xi \in \Gamma(\ker{\pi_*}^\perp)$. Then,

$$\begin{aligned}\nabla_\beta\xi &= \phi\nabla_\beta\phi\xi \\ &= \phi(\nabla_\beta\mathscr{T}\xi + \nabla_\beta N\xi) \\ &= \phi(\mathcal{A}_\beta(\mathscr{T}\xi) + \hat{\nabla}_\beta\mathscr{T}\xi + \mathcal{A}_\beta(N\xi) + \mathcal{H}\,\nabla_\beta N\xi) \\ &= (\mathscr{T}\mathcal{A}_\beta(\mathscr{T}\xi) + t\hat{\nabla}_\beta\mathscr{T}\xi + t\mathcal{A}_\beta(N\xi) + \mathscr{T}\,\mathcal{H}\,\nabla_\beta N\xi) \\ &\quad + (N\mathcal{A}_\beta(\mathscr{T}\xi) + n\hat{\nabla}_\beta\mathscr{T}\xi + n\mathcal{A}_\beta(N\xi) + N\,\mathcal{H}\,\nabla_\beta N\xi).\end{aligned}$$

Therefore:

$$\begin{aligned}\mathcal{H}\,\nabla_\beta\xi &= N\mathcal{A}_\beta(\mathscr{T}\xi) + n\hat{\nabla}_\beta\mathscr{T}\xi + n\mathcal{A}_\beta(N\xi) + N\,\mathcal{H}\,\nabla_\beta N\xi, \\ \hat{\nabla}_\beta\xi &= \mathscr{T}\mathcal{A}_\beta(\mathscr{T}\xi) + t\hat{\nabla}_\beta\mathscr{T}\xi + t\mathcal{A}_\beta(N\xi) + \mathscr{T}\,\mathcal{H}\,\nabla_\beta N\xi.\end{aligned}$$

By lemma 5.1, $\ker{\pi_*}^\perp$ is totally geodesic if and only if $\hat{\nabla}_\beta\xi = 0$ for each $\beta, \xi \in \Gamma(\ker{\pi_*}^\perp)$, and thus the result follows in a similar manner as theorem 5.4. □

We now combine these two theorems. It is known in the literature that, if $\ker\pi_*$ and $\ker{\pi_*}^\perp$ define totally geodesic foliations on M, then M can be written as the product of the integral manifolds of $\ker\pi_*$ and $\ker{\pi_*}^\perp$, denoted $M_{\ker\pi_*}$ and $M_{\ker{\pi_*}^\perp}$ respectively. Therefore, the previous results can be used to provide conditions under which the total manifold M can be written as a product manifold. This idea is summarized in the following result.

Corollary 5.6. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Then, if at least one of the conditions in theorem 5.4 and at least one in theorem 5.5 are verified for $\ker\pi_*$ and $\ker{\pi_*}^\perp$ respectively, then it must be the case that*

$$M = M_{\ker\pi_*} \times M_{\ker{\pi_*}^\perp}.$$

That is, M can be thought of as a product manifold of the integral manifolds of $\ker\pi_*$ and $\ker{\pi_*}^\perp$.

Moving on from the distributions $\ker\pi_*$ and $\ker{\pi_*}^\perp$, we develop equivalent conditions for $\mathcal{D}^{\theta_j}$ to be totally geodesic.

Theorem 5.7. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$ where $\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$. Then the following are equivalent: for all $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$ and $U \in \Gamma(\mathcal{D}^{\theta_i})$, where $i \neq j$ and $i, j \in \{1, 2\}$:*

I) $\mathcal{D}^{\theta_j}$ *is totally geodesic,*

II) *i)* $0 = g(X, \mathcal{T}_Y(ntU) + \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)),$

ii) $0 = \bar{g}(\pi_*(ntU), (\nabla\pi_*)(Y, X)) + \bar{g}(\pi_*(nU), (\nabla\pi_*)(\phi Y, X)),$

III) *i)* $0 = g(\hat{\nabla}_X tY, tU) + g(X, \mathcal{T}_{tU}(nY) + \mathcal{T}_Y(ntU))$,

ii) $0 = g(\hat{\nabla}_X tY, tU) + \bar{g}((\nabla\pi_*)(tU, X), \pi_*(nY)) + \bar{g}((\nabla\pi_*)(Y, X), \pi_*(ntU))$,

IV) $0 = \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) - \bar{g}(\pi_*(nU), (\nabla\pi_*)(X, tY)) + g(\phi U, \hat{\nabla}_X tY + \mathcal{A}_{nY}(X))$,

V) $0 = \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) + \bar{g}(\pi_*(\phi\nabla_X Y), \pi_*(nU)) + g(\nabla_X tY, tU)$.

VI) $\mathscr{T}\mathcal{T}_X(tY) + t\hat{\nabla}_X tY + t\mathcal{T}_X(nY) + \mathscr{T}(\mathcal{A}_{nY}(X))$ *has no components in* $\mathcal{D}^{\theta_i} \oplus \ker\pi_*{}^{\perp}$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$ and $U \in \Gamma(\mathcal{D}^{\theta_i})$. Then, from theorem 4.2,

$$\begin{aligned} g(\nabla_X Y, U) &= -\csc^2\theta_i \Big[g(X, \mathcal{T}_Y(ntU) + \mathcal{T}_{tY}(nU) + \mathcal{A}_{nY}(nU)) \Big] \\ &= \csc^2\theta_i \Big[g(\mathcal{T}_Y(X), ntU) + g(nU, \mathcal{T}_{tY}(X) + \mathcal{A}_{nY}(X)) \Big] \\ &= \csc^2\theta_i \Big[\bar{g}(\pi_*(\mathcal{T}_Y(X)), \pi_*(ntU)) + \bar{g}(\pi_*(nU), \pi_*(\mathcal{T}_{tY}(X) + \mathcal{A}_{nY}(X))) \Big] \\ &= \csc^2\theta_i \Big[\bar{g}(\pi_*(\nabla_Y X), \pi_*(ntU)) + \bar{g}(\pi_*(nU), \pi_*(\nabla_{\phi Y} X) \Big] \\ &= -\csc^2\theta_i \Big[\bar{g}((\nabla\pi_*)(Y, X), \pi_*(ntU)) + \bar{g}(\pi_*(nU), (\nabla\pi_*)(\phi Y, X) \Big] \end{aligned}$$

We know from lemma 5.1 that $\mathcal{D}^{\theta_j}$ is totally geodesic if and only if $g(\nabla_X Y, U) = 0$ for each $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$ and $U \in \Gamma(\mathcal{D}^{\theta_i})$. Therefore, the first and last equations above show that item I), item II(i), and item II(ii) are equivalent.

To prove the equivalence of item I), item III(i), and item III(ii), we use theorem 4.2 to write the following:

$$\begin{aligned} g(\nabla_X Y, U) &= \sec^2\theta_i \Big[g(\hat{\nabla}_X tY, tU) + g(X, \mathcal{T}_{tU}(nY) + \mathcal{T}_Y(ntU)) \Big] \\ &= \sec^2\theta_i \Big[g(\hat{\nabla}_X tY, tU) - g(\mathcal{T}_{tU}(X), nY) - g(\mathcal{T}_Y(X), ntU) \Big] \\ &= \sec^2\theta_i \Big[g(\hat{\nabla}_X tY, tU) - \bar{g}(\pi_*(\mathcal{T}_{tU}(X)), \pi_*(nY)) - \bar{g}(\pi_*(\mathcal{T}_Y(X)), \pi_*(ntU)) \Big] \\ &= \sec^2\theta_i \Big[g(\hat{\nabla}_X tY, tU) - \bar{g}(\pi_*(\nabla_{tU} X), \pi_*(nY)) - \bar{g}(\pi_*(\nabla_Y X), \pi_*(ntU)) \Big] \\ &= \sec^2\theta_i \Big[g(\hat{\nabla}_X tY, tU) + \bar{g}((\nabla\pi_*)(tU, X), \pi_*(nY)) + \bar{g}((\nabla\pi_*)(Y, X), \pi_*(ntU)) \Big] \end{aligned}$$

Using lemma 5.1 again, the first and last equations above show the equivalence of item I), item III(i), and item III(ii).

To prove equivalence of item I) and item IV), we expand $g(\nabla_X Y, U)$ in the following manner:

$$\begin{aligned} g(\nabla_X Y, U) &= g(\nabla_X \phi Y, \phi U) \\ &= g(\nabla_X tY + \nabla_X nY, tU + nU) \\ &= g(\mathcal{T}_X(tY) + \hat{\nabla}_X tY + \mathcal{T}_X(nY) + \mathcal{H}\nabla_X nY, tU + nU) \\ &= g(\hat{\nabla}_X tY, tU) + g(\mathcal{T}_X(nY), tU) + g(\mathcal{T}_X(tY), nU) + g(\mathcal{H}\nabla_X nY, nU) \\ &= g(\hat{\nabla}_X tY, \phi U) - g(nY, \mathcal{T}_X(tU)) + g(\mathcal{T}_X(tY), nU) + g(\mathcal{H}\nabla_X nY, \phi U) \\ &= g(\phi U, \hat{\nabla}_X tY + \mathcal{H}\nabla_X nY) - \bar{g}(\pi_*(nY), \pi_*(\mathcal{T}_X(tU))) + \bar{g}(\pi_*(nU), \pi_*(\mathcal{T}_X(tY))) \\ &= g(\phi U, \hat{\nabla}_X tY + \mathcal{A}_{nY}(X)) + \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) - \bar{g}(\pi_*(nU), (\nabla\pi_*)(X, tY)). \end{aligned}$$

By lemma 5.1, the final equation above demonstrates that item I) and item IV) are equivalent.

We now employ a similar method to show the equivalency of item I) and item V), this time employing item I(i) from corollary 3.5:

$$\begin{aligned}
g(\nabla_X Y, U) &= g(\phi \nabla_X Y, \phi U) \\
&= g(\phi(\mathcal{T}_X(Y) + \hat{\nabla}_X Y), tU + nU) \\
&= g(\mathscr{T}\mathcal{T}_X(Y) + N\mathcal{T}_X(Y) + t\hat{\nabla}_X Y + n\hat{\nabla}_X Y, tU + nU) \\
&= g(\mathscr{T}\mathcal{T}_X(Y) + t\hat{\nabla}_X Y, tU) + g(N\mathcal{T}_X(Y) + n\hat{\nabla}_X Y, nU) \\
&= g(\hat{\nabla}_X tY + \mathcal{T}_X(nY), tU) + g(N\mathcal{T}_X(Y) + n\hat{\nabla}_X Y, nU) \\
&= g(\hat{\nabla}_X tY, tU) - g(nY, \mathcal{T}_X(tU)) + g(N\mathcal{T}_X(Y) + n\hat{\nabla}_X Y, nU) \\
&= g(\hat{\nabla}_X tY, tU) - \bar{g}(\pi_*(nY), \pi_*(\mathcal{T}_X(tU))) + \bar{g}(\pi_*(N\mathcal{T}_X(Y) + n\hat{\nabla}_X Y), \pi_*(nU)) \\
&= g(\hat{\nabla}_X tY, tU) + \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) + \bar{g}(\pi_*(\phi(\mathcal{T}_X(Y) + \hat{\nabla}_X Y)), \pi_*(nU)) \\
&= g(\hat{\nabla}_X tY, tU) + \bar{g}(\pi_*(nY), (\nabla\pi_*)(X, tU)) + \bar{g}(\pi_*(\phi\nabla_X Y), \pi_*(nU)).
\end{aligned}$$

Using lemma 5.1 again, the result follows.

For the final result, we quote eq. (5.1):

$$\hat{\nabla}_X Y = \mathscr{T}\mathcal{T}_X(tY) + t\hat{\nabla}_X tY + t\mathcal{T}_X(nY) + \mathscr{T}(\mathcal{A}_{nY}(X)).$$

Using lemma 5.1 once more, it is clear item I) and item VI) are equivalent. □

Using this theorem above, we may find equivalent conditions to express the integral manifold of $\ker \pi_*$ (denoted $M_{\ker \pi_*}$) as a product manifold, in a similar manner to corollary 5.6. This time, though, $M_{\ker \pi_*}$ is broken down into the integral manifolds of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, denoted $M_{\mathcal{D}^{\theta_1}}$ and $M_{\mathcal{D}^{\theta_2}}$ respectively.

Corollary 5.8. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Then, if at least one condition in theorem 5.7 is verified for both $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$, it must be the case that*

$$M_{\ker \pi_*} = M_{\mathcal{D}^{\theta_1}} \times M_{\mathcal{D}^{\theta_2}}.$$

That is, $M_{\ker \pi_*}$ can be thought of as a product manifold of the integral manifolds of $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$.

To finish this section, we incorporate the totally geodesic condition in relating the canonical forms $t, n, \mathscr{T}$, and N. As shown before in theorem 3.8, $\nabla n = 0$ if and only if $\nabla\mathscr{T} = 0$; from this, it seems natural to try and find a relationship between the parallel conditions for N and t. Interestingly, these two forms are related in a similar manner when we incorporate total geodesicity. The following theorem summarizes this result.

Theorem 5.9. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to Riemannian manifold $(N, \bar{g})$. If $\ker \pi_*$ is totally geodesic, then $\nabla N = 0$ if and only if $\nabla t = 0$.*

Proof. Assume $\ker \pi_*$ is totally geodesic and let $U, V, X \in \Gamma(\ker \pi_*)$. Then:

$$\begin{aligned}
g(X, \mathscr{T}\mathcal{T}_U(V)) = g(X, \phi\mathcal{T}_U(V)) = g(\phi X, \mathcal{T}_U(V)) = g(nX, \mathcal{T}_U(V)) = -g(\mathcal{T}_U(nX), V) = -g(\phi\mathcal{T}_U(nX), \phi V) \\
= -g(t\mathcal{T}_U(nX), tV) - g(n\mathcal{T}_U(nX), nV).
\end{aligned}$$

By assumption that $\nabla N = 0$, we then know $n\mathcal{T}_U(nX) = \mathcal{T}_U(\mathscr{T}nX)$ by eq. (3.7). Thus, by corollary 3.4

$$\begin{aligned}
g(X, \mathscr{T}\mathcal{T}_U(V)) &= -g(t\mathcal{T}_U(nX), tV) - g(\mathcal{T}_U(\mathscr{T}nX), nV) \\
&= -g(t\mathcal{T}_U(nX), tV) + g(\mathscr{T}nX, \mathcal{T}_U(nV)) \\
&= -g(t\mathcal{T}_U(nX), tV) + g(X, \mathcal{T}_U(nV)) - g(t^2X, \mathcal{T}_U(nV)) \\
&= -g(\mathcal{T}_U(nX), t^2V) + g(X, \mathcal{T}_U(nV)) + g(\mathcal{T}_U(t^2X), nV) \\
&= g(\mathcal{T}_U(t^2V), nX) + g(X, \mathcal{T}_U(nV)) + g(\mathcal{T}_U(t^2X), nV) \\
\implies g(X, \mathscr{T}\mathcal{T}_U(V) - \mathcal{T}_U(nV)) &= g(\mathcal{T}_U(t^2V), nX) + g(\mathcal{T}_U(t^2X), nV).
\end{aligned}$$

However, since $\ker \pi_*$ is totally geodesic, we know by corollary 5.3 that $\mathcal{T}_U(t^2V) = 0 = \mathcal{T}_U(t^2X)$, implying $g(X, \mathscr{T}\mathcal{T}_U(V) - \mathcal{T}_U(nV)) = 0$. Since this equation is true for all $X, U, V \in \Gamma(\ker \pi_*)$, we know $\mathscr{T}\mathcal{T}_U(V) - \mathcal{T}_U(nV) = 0$, meaning $\nabla t = 0$ by eq. (3.4).

To show the converse, let $\beta \in \Gamma(\ker \pi_*{}^{\perp})$. Then:

$$\begin{aligned} g(\beta, n\mathcal{T}_U(\xi)) &= g(\beta, \phi\mathcal{T}_U(\xi)) = g(\phi\beta, \mathcal{T}_U(\xi)) = g(\mathscr{T}\beta, \mathcal{T}_U(\xi)) = -g(\mathcal{T}_U(\mathscr{T}\beta), \xi) = -g(\phi\mathcal{T}_U(\mathscr{T}\beta), \phi\xi) \\ &= -g(\mathscr{T}\mathcal{T}_U(\mathscr{T}\beta), \mathscr{T}\xi) - g(N\mathcal{T}_U(\mathscr{T}\beta), N\xi). \end{aligned}$$

Assuming $\nabla t = 0$, we know $\mathscr{T}\mathcal{T}_U(\mathscr{T}\beta) = \mathcal{T}_U(n\mathscr{T}\beta)$ by eq. (3.4). Then by corollary 3.4,

$$\begin{aligned} g(\beta, n\mathcal{T}_U(\xi)) &= -g(\mathcal{T}_U(n\mathscr{T}\beta), \mathscr{T}\xi) - g(N\mathcal{T}_U(\mathscr{T}\beta), N\xi) \\ &= g(n\mathscr{T}\beta, \mathcal{T}_U(\mathscr{T}\xi)) - g(N\mathcal{T}_U(\mathscr{T}\beta), N\xi) \\ &= g(\beta, \mathcal{T}_U(\mathscr{T}\xi)) - g(N^2\beta, \mathcal{T}_U(\mathscr{T}\xi)) - g(N\mathcal{T}_U(\mathscr{T}\beta), N\xi) \\ &= g(\beta, \mathcal{T}_U(\mathscr{T}\xi)) - g(\mathcal{T}_U(\mathscr{T}\xi), N^2\beta) - g(\mathcal{T}_U(\mathscr{T}\beta), N^2\xi) \\ \implies g(\beta, n\mathcal{T}_U(\xi) - \mathcal{T}_U(\mathscr{T}\xi)) &= -g(\mathcal{T}_U(\mathscr{T}\xi), N^2\beta) - g(\mathcal{T}_U(\mathscr{T}\beta), N^2\xi). \end{aligned}$$

Due to $\ker \pi_*$ being totally geodesic, we see $\mathcal{T}_U(\mathscr{T}\xi) = 0 = \mathcal{T}_U(\mathscr{T}\beta)$, implying $g(\beta, n\mathcal{T}_U(\xi) - \mathcal{T}_U(\mathscr{T}\xi)) = 0$. Therefore, using eq. (3.7), we see $(\nabla_U N)\xi = n\mathcal{T}_U(\xi) - \mathcal{T}_U(\mathscr{T}\xi) = 0$ for all $U \in \Gamma(\ker \pi_*)$ and $\xi \in \Gamma(\ker \pi_*{}^{\perp})$, implying $\nabla N = 0$. □

6. Pluriharmonicity

We now introduce the pluriharmonic condition for the submersion π. Pluriharmonic morphisms are of interest because they are a generalization of harmonic morphisms, and these maps are used widely in theoretical differential geometry and many applied mathematical fields like Quantum Field Theory, gravitation in Astrophysics, Geophysics, and more. In this report, we use the pluriharmonic condition to derive equivalent conditions for integrability and special types of geodesicity for the pointwise slant distributions $\mathcal{D}^{\theta_j}$, and we also develop equivalent conditions for pluriharmonicity in special cases. First, we define what it means for the submersion π to be pluriharmonic, $n\mathcal{D}^{\theta_j}$-geodesic, and totally geodesic (not to be confused with a totally geodesic *distribution*, as covered in the previous section), then present relevant results.

Definition 6.1. Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Suppose that S is a distribution on M. We say that π is $S - \phi - pluriharmonic$ if for each $X, Y \in \Gamma(S)$

$$(\nabla\pi_*)(X, Y) + (\nabla\pi_*)(\phi X, \phi Y) = 0.$$

If S is given by the tangent bundle TM, then π is said to be $\phi - pluriharmonic$.

Definition 6.2. Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. We say that π is a *totally geodesic map* if for each $X, Y \in \Gamma(TM)$

$$(\nabla\pi_*)(X, Y) = 0.$$

We also say π is an $n\mathcal{D}^{\theta_j}$*-geodesic map* if, for all $W, Z \in \Gamma(\mathcal{D}^{\theta_j})$,

$$(\nabla\pi_*)(nW, nZ) = 0.$$

From the definition of a pluriharmonic submersion, we have two immediate corollaries.

Corollary 6.3. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. If π is $\phi-$pluriharmonic, then:*

$$(\nabla\pi_*)(\phi X, Y) = -(\nabla\pi_*)(X, \phi Y)$$

Proof. Because π is $\phi-$pluriharmonic we have that

$$(\nabla\pi_*)(X, Y) = -(\nabla\pi_*)(\phi X, \phi Y)$$

for each $X, Y \in \Gamma(TM)$. But then it must be the case that:

$$(\nabla\pi_*)(\phi X, Y) = -(\nabla\pi_*)(\phi^2 X, \phi Y) = -(\nabla\pi_*)(X, \phi Y)$$

□

Corollary 6.4. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Then for each $X, Y \in \Gamma(\ker \pi_*)$,*

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(Y, X).$$

Proof. Let $X, Y \in \Gamma(\ker \pi_*)$. Then:

$$\begin{aligned}(\nabla\pi_*)(X,Y) - (\nabla\pi_*)(Y,X) &= \nabla^\pi_X \pi_*(Y) - \pi_*(\nabla_X Y) - (\nabla^\pi_Y \pi_*(X) - \pi_*(\nabla_Y X))\\ &= \pi_*(\nabla_Y X - \nabla_X Y)\\ &= \pi_*([Y,X]).\end{aligned}$$

Because both X and Y are vertical, we know their Lie bracket is vertical. Therefore, their push forward is 0, and the desired result follows. □

Now that these corollaries are established, we may now present more relevant results using these corollaries. First, we show equivalent conditions for π being pluriharmonic using two geodesic conditions.

Proposition 6.5. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$, and assume that $\ker \pi_*$ is totally geodesic and that π is an $n\mathcal{D}^{\theta_j}$-geodesic map, where $j \in \{1, 2\}$. Then the following are equivalent:*

I) π is $\mathcal{D}^{\theta_j} - \phi-$pluriharmonic,

II) $\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$ for each $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$,

III) $n\nabla_{tX}Y + N\mathcal{A}_{nX}(Y) + n\hat{\nabla}_{nX}Y - \mathcal{H}\nabla_{nX}nY = 0$ for each $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$,

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$. Then:

$$\begin{aligned}(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X,\phi Y) &= \nabla^\pi_X \pi_*(Y) - \pi_*(\nabla_X Y) + \nabla^\pi_{\phi X}\pi_*(\phi Y) - \pi_*(\nabla_{\phi X}\phi Y)\\ &= -\pi_*(\nabla_X Y) + \nabla^\pi_{nX}\pi_*(nY) - \pi_*(\nabla_{\phi X}\phi Y)\\ &= -\pi_*(\mathcal{T}_X(Y)) + \nabla^\pi_{nX}\pi_*(nY)\\ &\quad - \pi_*(\nabla_{tX}tY + \nabla_{tX}nY + \nabla_{nX}tY + \nabla_{nX}nY)\\ &= -\pi_*(\mathcal{T}_X(Y)) + (\nabla\pi_*)(nX,nY)\\ &\quad - \pi_*(\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY)).\end{aligned} \tag{6.1}$$

Using the condition that $\ker \pi_*$ is totally geodesic, we know from corollary 5.3 that $\pi_*(\mathcal{T}_X(Y)) = \pi_*(0) = 0$ and that $\pi_*(\mathcal{T}_{tX}(tY)) = 0$. We also know, since π is $n\mathcal{D}^{\theta_j}-$ geodesic, that $(\nabla\pi_*)(nX, nY) = 0$. Then:

$$(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X,\phi Y) = -\pi_*(\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY)).$$

Thus, if π is $\mathcal{D}^{\theta_j} - \phi-$pluriharmonic, then $\pi_*(\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY)) = 0$. Since $\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY)$ is horizontal, then it must be that $\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$. Conversely, if $\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$, then $(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X, \phi Y) = 0$, proving item I) is equivalent to item II).

For the other result, we start at a similar point but use the parallel condition on ϕ to write the following:

$$\begin{aligned}(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X,\phi Y) &= \nabla^\pi_X \pi_*(Y) - \pi_*(\nabla_X Y) + \nabla^\pi_{\phi X}\pi_*(\phi Y) - \pi_*(\nabla_{\phi X}\phi Y)\\ &= -\pi_*(\nabla_X Y) + \nabla^\pi_{nX}\pi_*(nY) - \pi_*(\phi(\nabla_{\phi X}Y))\\ &= -\pi_*(\nabla_X Y) + \nabla^\pi_{nX}\pi_*(nY)\\ &\quad - \pi_*(\phi(\mathcal{T}_{tX}(Y) + \hat{\nabla}_{tX}Y + \mathcal{A}_{nX}(Y) + \hat{\nabla}_{nX}Y))\\ &= -\pi_*(\mathcal{T}_X(Y)) + \nabla^\pi_{nX}\pi_*(nY)\\ &\quad - \pi_*(N\mathcal{T}_{tX}(Y) + n\hat{\nabla}_{tX}Y + N\mathcal{A}_{nX}(Y) + n\hat{\nabla}_{nX}Y)\end{aligned}$$

Using the condition that $\ker \pi_*$ is totally geodesic, we know that $\pi_*(\mathcal{T}_X(Y)) = 0$, $N\mathcal{T}_{tX}(Y) = N(0) = 0$, and that $\hat{\nabla}_{tX}Y = \nabla_{tX}Y$. Moreover, since π is $n\mathcal{D}^{\theta_j}-$ geodesic, we know $\nabla^\pi_{nX}\pi_*(nY) = \pi_*(\nabla_{nX}nY)$. Therefore,

$$(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X,\phi Y) = -\pi_*(n\nabla_{tX}Y + N\mathcal{A}_{nX}(Y) + n\hat{\nabla}_{nX}Y - \mathcal{H}\nabla_{nX}nY).$$

The equivalence of item I) and item III) then follows in a similar manner. □

Proposition 6.6. *Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Let $j \in \{1, 2\}$ and suppose that π is $\mathcal{D}^{\theta_j} - \phi -$ pluriharmonic. Then the following are equivalent:*

I) π is $n\mathcal{D}^{\theta_j}$-geodesic, *II)* $\mathcal{T}_X(Y) + \mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$ *for all* $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$*.*

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$. By assumption, $(\nabla\pi_*)(X,Y) + (\nabla\pi_*)(\phi X, \phi Y) = 0$, and then by eq. (6.1):

$$\begin{aligned} 0 =& -\pi_*(\mathcal{T}_X(Y)) + (\nabla\pi_*)(nX, nY) - \pi_*(\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY)) \\ \implies (\nabla\pi_*)(nX, nY) =& \pi_*(\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) + \mathcal{T}_X(Y)). \end{aligned}$$

Since all terms in the argument of the push-forward of π are horizontal, the result is then clear. □

Finally, we may update the integrability condition given in theorem 4.3 for the pointwise slant distributions $\mathcal{D}^{\theta_i}$ when our submersion is $\phi-$pluriharmonic. Only the changed conditions are listed below, but all others remain true.

Proposition 6.7. *Let* π *be a pointwise bi-slant Riemannian submersion from locally product Riemannian manifold* (M, ϕ, g) *to Riemannian manifold* $(N, \bar{g})$ *where* $\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$*. Let* $i, j \in \{1, 2\}$ *where* $i \neq j$*, and assume* π *is* $\phi-$*pluriharmonic. Then the following conditions are equivalent:*

I) $\mathcal{D}^{\theta_i}$ *is integrable,*

II) $0 = \bar{g}(\pi_*(nU), 2(\nabla\pi_*)(Y, \phi X) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(Y, nX))$ *for each* $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ *and* $U \in \Gamma(\mathcal{D}^{\theta_j})$*.*

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$. By theorem 4.3, we know $\mathcal{D}^{\theta_i}$ is integrable if and only if $0 = \bar{g}(\pi_*(nU), (\nabla\pi_*)(tX, Y) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(tY, X) - (\nabla\pi_*)(nY, X))$. We may then manipulate the metric using the pluriharmonicity condition and corollary 6.3:

$$\begin{aligned} &\bar{g}(\pi_*(nU), (\nabla\pi_*)(tX, Y) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(tY, X) - (\nabla\pi_*)(nY, X)) \\ &= \bar{g}(\pi_*(nU), (\nabla\pi_*)(Y, tX) - (\nabla\pi_*)(tY, X) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(nY, X)) \\ &= \bar{g}(\pi_*(nU), ((\nabla\pi_*)(Y, \phi X) - (\nabla\pi_*)(Y, nX)) - ((\nabla\pi_*)(\phi Y, X) - (\nabla\pi_*)(nY, X)) \\ &\quad + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(nY, X)) \\ &= \bar{g}(\pi_*(nU), 2(\nabla\pi_*)(Y, \phi X) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(Y, nX)) \end{aligned}$$

The result then follows. □

7. ϕ-INVARIANCE

For the final section of this report, we discuss ϕ-invariance. This condition seems fairly similar to the pluriharmonic case, and while its mathematical implementation is quite similar, the application and interpretation is not. As we will see, many results are similar (or even remain the same) when compared to the pluriharmonic case, but certain simplifications exist in important integrability conditions under this assumption. Thus, let us define precisely the ϕ-invariant condition, and then provide relevant results.

Definition 7.1. Let π be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold (M, ϕ, g) to a Riemannian manifold $(N, \bar{g})$. Suppose that S is a distribution on M. We say that π is $S - \phi -$ *invariant* if for each $X, Y \in \Gamma(S)$

$$(\nabla\pi_*)(X, Y) = (\nabla\pi_*)(\phi X, \phi Y)$$

If S is given by the tangent bundle TM, then π is said to be $\phi -$ *invariant*.

Clearly, if π is $\phi-$invariant, then $(\nabla\pi_*)(\phi X, Y) = (\nabla\pi_*)(X, \phi Y)$ for any vector fields X and Y. Moreover, as said before, certain theorems originally related to $\phi-$pluriharmonicity remain the same when considering $\phi-$invariance instead. The next proposition is one such case, which has the same results as proposition 6.5.

Proposition 7.2. *Let* π *be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold* (M, ϕ, g) *to a Riemannian manifold* $(N, \bar{g})$*, and assume that* $\ker\pi_*$ *is totally geodesic and that* π *is an* $n\mathcal{D}^{\theta_j}$*-geodesic map, where* $j \in \{1, 2\}$*. Then the following are equivalent:*

I) π *is* $\mathcal{D}^{\theta_j} - \phi-$*invariant,*

II) $\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$ *for each* $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$*,* *III)* $n\nabla_{tX}Y + N\mathcal{A}_{nX}(Y) + n\hat{\nabla}_{nX}Y - \mathcal{H}\nabla_{nX}nY = 0$ *for each* $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$,

Proof. This follows in a similar manner to the proof of proposition 6.5. □

While the previous proposition remained invariant in comparison to a corresponding result under pluriharmonicity, the next proposition demonstrates that corresponding results by no means have to remain the same when converting to $\phi-$invariance from $\phi-$pluriharmonicity.

Proposition 7.3. *Let* π *be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold* (M, ϕ, g) *to a Riemannian manifold* $(N, \bar{g})$. *Let* $j \in \{1, 2\}$ *and suppose that* π *is* $\mathcal{D}^{\theta_j} - \phi-$*invariant. Then the following are equivalent:*

I) π *is* $n\mathcal{D}^{\theta_j}-$ *geodesic,*

II) $\mathcal{T}_{tX}(tY) - \mathcal{T}_X(Y) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) = 0$ *for all* $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}^{\theta_j})$ and assume π is $\mathcal{D}^{\theta_j} - \phi-$invariant. Then, in a similar manner to eq. (6.1), we know the following is true:

$$0 = (\nabla\pi_*)(X, Y) - (\nabla\pi_*)(\phi X, \phi Y)$$
$$0 = -\pi_*(\mathcal{T}_X(Y)) - (\nabla\pi_*)(nX, nY) + \pi_*(\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY))$$
$$\implies (\nabla\pi_*)(nX, nY) = \pi_*(\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) - \mathcal{T}_X(Y)). \tag{7.1}$$

Since $\mathcal{T}_{tX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) - \mathcal{T}_X(Y)$ is horizontal, the result follows. □

We also may propose equivalent conditions for $\ker\pi_*$ to be totally geodesic under the $\phi-$invariant condition.

Proposition 7.4. *Let* π *be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold* (M, g, ϕ) *to a Riemannian manifold* $(N, \bar{g})$. *Suppose that* π *is* $\ker\pi_* - \phi -$ *invariant. Then the following are equivalent: for all* $X, Y \in \Gamma(\ker\pi_*)$,

I) $\ker\pi_*$ *defines totally geodesic foliations on* M

II) $(\nabla\pi_*)(nX, nY) = \pi_*(\mathcal{A}_{nX}(tY) + \mathcal{A}_{nY}(tX) + \mathcal{T}_{tX}(tY))$

Proof. Let $X, Y \in \Gamma(\ker\pi_*)$ and assume π is $\ker\pi_* - \phi-$ invariant. Then, in a similar manner to eq. (7.1),

$$(\nabla\pi_*)(nX, nY) = \pi_*(\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) + \mathcal{T}_{tX}(tY) - \mathcal{T}_X(Y)).$$

We see if $(\nabla\pi_*)(nX, nY) = \pi_*(\mathcal{A}_{nY}(tX) + \mathcal{A}_{nX}(tY) + \mathcal{T}_{tX}(tY))$ for all $X, Y \in \Gamma(\ker\pi_*)$, then $0 = \pi_*(\mathcal{T}_X(Y))$, implying $\mathcal{T}_X(Y) = 0$ since $\mathcal{T}_X(Y)$ is horizontal. Therefore, $\mathcal{T} = 0$ on $\ker\pi_*$, implying $\ker\pi_*$ is totally geodesic by corollary 5.3. The converse is clearly true. □

Lastly, we may present equivalent conditions for the integrability of $\mathcal{D}^{\theta_j}$. This condition is quite similar to the one given in proposition 6.7 but, interestingly, the condition simplifies nicely in comparison to the $\phi-$pluriharmonic case.

Proposition 7.5. *Let* π *be a pointwise bi-slant Riemannian submersion from a locally product Riemannian manifold* (M, ϕ, g) *to a Riemannian manifold* $(N, \bar{g})$ *where* $\ker\pi_* = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2}$. *Let* $i, j \in \{1, 2\}$ *where* $i \neq j$, *and assume* π *is* $\phi-$*invariant. Then the following conditions are equivalent:*

I) $\mathcal{D}^{\theta_i}$ *is integrable,*

II) $0 = \bar{g}(\pi_*(nU), (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(Y, nX))$ *for each* $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ *and* $U \in \Gamma(\mathcal{D}^{\theta_j})$

Proof. Again by theorem 4.3, we know $\mathcal{D}^{\theta_i}$ is integrable if and only if $0 = \bar{g}(\pi_*(nU), (\nabla\pi_*)(tX, Y) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(tY, X) - (\nabla\pi_*)(nY, X))$ for each $X, Y \in \Gamma(\mathcal{D}^{\theta_i})$ and $U \in \Gamma(\mathcal{D}^{\theta_j})$. We may then manipulate this metric relation in a similar manner to the proof of proposition 6.7:

$$\begin{aligned}
&\bar{g}(\pi_*(nU), (\nabla\pi_*)(tX, Y) + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(tY, X) - (\nabla\pi_*)(nY, X)) \\
&= \bar{g}(\pi_*(nU), ((\nabla\pi_*)(Y, \phi X) - (\nabla\pi_*)(Y, nX)) - ((\nabla\pi_*)(\phi Y, X) - (\nabla\pi_*)(nY, X)) \\
&\quad + (\nabla\pi_*)(nX, Y) - (\nabla\pi_*)(nY, X)) \\
&= \bar{g}(\pi_*(nU), -(\nabla\pi_*)(Y, nX) + (\nabla\pi_*)(nX, Y))
\end{aligned}$$

The result then follows. □

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